Generating functions and sum rules for quantum oscillator

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Generating functions and sum rules are discussed for transition probabilities between quantum oscillator eigenstates with time-dependent parameters.

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The approach supposed by Feynman [1] for disentangling of non-commuting operators (FDM) was successfully applied to various time-dependent problems in quantum mechanics [2]–[5]. Using FDM one can [1, 3] calculate the transition probabilities between eigenstates $|m, t \to -\infty\rangle$ and $|n, t \to +\infty\rangle$ of a quantum oscillator with definite initial and final quantum numbers, and compute corresponding generating functions. If the oscillator has a constant frequency ω being under an external force $f(t)$[1], for the generating function one has[2]:

$$G(u, v \mid \nu) = \sum_{m,n=0}^{\infty} w_{mn}(\nu) u^m v^n = (1 - uv)^{-1} \exp\left\{ -\nu \, \frac{(1-u)(1-v)}{1-uv} \right\}, \tag{1}$$

where u, v are the auxiliary complex numbers and ν is the oscillator excitation parameter:

$$\nu = \frac{1}{2\omega} \left| \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \right|^2, \quad \hbar = m = 1. \tag{2}$$

From (1) one obtains some non-trivial relations for probabilities w_{mn}. First, integrating by ν, one can derive several peculiar sum rules

$$\begin{gathered} \int_0^{\infty} w_{mn}(\nu) d\nu = 1, \quad \langle \nu \rangle_{mn} = \int_0^{\infty} w_{mn}(\nu) \nu d\nu = m + n + 1, \\ \langle \Delta\nu^2 \rangle_{mn} = \int_0^{\infty} w_{mn}(\nu) \left[\nu - \langle \nu \rangle_{mn} \right]^2 d\nu = 2mn + m + n + 1 \end{gathered} \tag{3}$$

for arbitrary quantum numbers m and n. For the initial vacuum state ($m = 0$) we have $\langle \Delta\nu^2 \rangle_{0n} = \langle \nu \rangle_{0n}$ which corresponds to the Poisson distribution.

Taking in (1) $u = v$ one has

$$G(u, u|\nu) = (1 - u^2)^{-1} \exp\left\{ 2\nu u (1 + u)^{-1} - \nu \right\}$$

which yields

$$S_k(\nu) \equiv \sum_{m+n=k} w_{mn}(\nu) = e^{-\nu} p_k(\nu), \tag{4}$$

where $p_k(\nu) = \sum_{s=0}^{k} (-1)^s L_s(2\nu)$ and L_s are the Laguerre polynomials, so

$$p_0 = 1, \quad p_1 = 2\nu, \quad p_2 = 2\nu^2 - 2\nu + 1, \quad p_3 = \frac{4}{3}\nu^3 - 4\nu^2 + 4\nu, \ \ldots$$

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[1] A natural condition $f(t) \to 0$ at $t \to \pm\infty$ is supposed.
[2] This expression can be easily derived [5] from Schwinger result [6] for a sum $\sum_{m=0}^{\infty} w_{mn} u^m$.

For the oscillator with variable frequency $\omega(t)$ without an external force[3] ($f(t) \equiv 0$) one obtains

$$G(u, v \mid \rho) = \sum_{m,n=0}^{\infty} w_{mn}(\rho) u^m v^n = \sqrt{(1-\rho)/\left[(1-uv)^2 - \rho(u-v)^2\right]}, \tag{5}$$

and

$$\int_0^1 G(u, v \mid \rho)(1-\rho)^{-1} d\rho = \frac{2}{u-v}(\operatorname{Arctanh} u - \operatorname{Arctanh} v). \tag{6}$$

Here ρ, $0 \le \rho \le 1$ is the oscillator excitation parameter; details can be found in [3, 5]. The frequency $\omega(t)$ is an arbitrary real time function. As usual, we propose the boundary conditions:

$$\omega(t) \to \omega_{\pm} \quad \text{at} \quad t \to \pm\infty$$

which allows one to define the final and initial eigenstates of the oscillator. Note that the expressions (1) and (5) result from the general Husimi expression [7].

As a result we have

$$\begin{aligned} &\int_0^1 \frac{w_{mn}(\rho)}{1-\rho} d\rho = \frac{1 + (-1)^{m+n}}{m+n+1}, \\ &\int_0^1 \frac{w_{mn}(\rho)}{\rho\sqrt{1-\rho}} d\rho = \frac{1 + (-1)^{m+n}}{\mid m-n \mid}, \quad m \neq n. \end{aligned} \tag{7}$$

Analogously to (3) let us calculate the integral $J_{mn} = \int_0^1 w_{mn}(\rho) d\rho$. For diagonal ($m = n$) transitions

$$J_{nn} = \frac{1}{2n+1}\left[1 + \frac{1}{(2n+3)(2n-1)}\right], \quad n = 0, 1, 2, ..., \tag{8}$$

at $m \neq n$ the expression for J_{mn} is more cumbersome.

Finally, taking in (5) $u = v$, one obtains

$$G(u, u \mid \rho) = \frac{\sqrt{1-\rho}}{1-u^2}$$

and

$$S_k(\rho) = \sum_{m+n=k} w_{mn}(\rho) = \begin{cases} \sqrt{1-\rho}, & k = 0, 2, 4, \ldots \\ 0, & k = 1, 3, 5, \ldots \end{cases} \tag{9}$$

(compare to (4)).

On differentiating subsequently the generating functions on parameters u and v, one can calculate the average quantum number in the final state

$$\left.\frac{\partial G}{\partial v}\right|_{v=1} = \sum_{m=0}^{\infty} \langle n \rangle_m u^m, \quad \langle n \rangle_m = \sum_{n=0}^{\infty} n w_{mn} = -\frac{1}{2} + \left(m + \frac{1}{2}\right)\frac{1+\rho}{1-\rho}, \tag{10}$$

the dispersion $\langle \Delta n^2 \rangle_m$ and other higher distribution momenta.

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[3] For this case the transitions occur only between states with same parity.

In [4] a more general model of a singular oscillator with variable frequency was considered:

$$\hat{H} = \frac{1}{2}p^2 + \frac{1}{2}\omega(t)^2 x^2 + \frac{g}{8x^2}, \\ 0 < x < +\infty, \quad g = \text{const}, \quad g > -1. \tag{11}$$

It is well known that at a fixed t the instantaneous spectrum of the Hamiltonian (11) is equidistant (see, e.g., [8]):

$$E_n = 2\omega(n - j), \qquad j = -\frac{1}{2} - \frac{1}{4}\sqrt{1+g}, \qquad n = 0, 1, 2, \ldots. \tag{12}$$

Using FDM, one can disentangle the operators in H and, after application of some group theory methods, calculate the transition amplitudes between initial $|m\rangle$ and final $|n\rangle$ states, being expressed in terms of the generalized Wigner function for the irreducible representation of the $su(1,1)$ algebra with weight j. As a result, one obtains explicit analytical expressions for the transition probabilities w_{mn}. The corresponding generating function takes the form:

$$g(u, v) = \sum_{m,n} w_{mn} u^m v^n = \frac{\lambda^{-2j}}{1 - uv \cdot \lambda^2} \tag{13}$$

where

$$\lambda = \frac{2(1-\rho)}{1 - \rho(u+v) + uv + \sqrt{\left[1 - \rho(u+v) + uv\right]^2 - 4uv(1-\rho)^2}}$$

(see [9] for details).

One can show that at $j = -3/4$, which corresponds to the regular oscillator ($g = 0$) with odd levels, these expressions turn into (5).

The most interesting physical case is the excitation of the oscillator ground level, i.e. $m = 0$. Taking in (13) $u = 0$ and, correspondingly, $\lambda = (1-\rho)/(1-\rho v)$, one obtains

$$g(0, v) = \sum_{n=0}^{\infty} w_{0n} v^n = \left(\frac{1-\rho}{1-\rho v}\right)^{-2j} \tag{14}$$

and

$$w_{0n}(\rho) = \frac{\Gamma(n-2j)}{n!\,\Gamma(-2j)} \rho^n (1-\rho)^{-2j}, \quad n = 0, 1, 2, \ldots \tag{15}$$

This formula also yields [3] the transition probabilities of the regular oscillator for $j = -1/4$ (even states, 0^{th} level excitation) and for $j = -3/4$ (odd states, 1^{th} level excitation).

To conclude, let us note that the generating function (13) determines also the adiabatic expansion of the probability w_{mn} for the case of small values of the excitation parameter. For example, for diagonal transitions ($m = n$) one has

$$w_{nn}(\rho) = 1 - 2[n^2 - (2n+1)j]\rho + \cdots = 1 - \frac{1}{2}\left(N^2 + N + 1\right)\rho + \mathcal{O}(\rho^2) \tag{16}$$

where

$$N = 2\sqrt{(n-j)^2 - (j(j+1) + 3/16)} - \frac{1}{2}.$$

In particular, $N = 2n$ for $j = -1/4$ and $N = 2n + 1$ for $j = -3/4$.

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